On Series of Iterated Linear Fractional Functions.[*]

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Introduction.

The two classes of power series (ascending and descending) are the most important series known to mathematical analysis. Recent investigations have brought to prominent notice another class of series which are also of prime importance; namely, the factorial series

$$a_0+\frac{a_1}{x}+\sum_{n=1}^{\infty}\frac{a_{n+1}n!}{x(x+1)(x+2)\ldots(x+n)}.$$

The simplicity and elegance of the general theory associated with these series is seen from the development of their fundamental properties by Landau.† Their importance throughout a large range of modern mathematical analysis is apparent from the demonstration by Watson‡ that most of the ordinary functions of analysis (which possess asymptotic expansions) actually are capable of being expressed in the form of convergent factorial series. An earlier result, of a character similar to this, is due to Horn,§ who has shown that the divergent Thomé normal series, which satisfy a linear differential equation with rational coefficients, may be transformed into convergent factorial series. That these series may be of great value in special problems is effectively illustrated in their recent use by Nörlund,‖ in his elegant paper on the integration of linear difference equations by means of factorial series.

With factorial series are to be associated the so-called binomial coefficient series

$$a_0+\sum_{n=1}^{\infty}a_n\frac{(x-1)(x-2)\ldots(x-n)}{n!},$$

the general theory of which has recently been developed by Landau¶ and others.

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*Presented to the American Mathematical Society, September, 1913.

† *Sitzungsber. d. Math.-Phys. Klasse d. Kgl. Bayer. Akad. d. Wiss.*, XXXVI (1906), pp. 151–218. See also the references in this paper.

‡ In a memoir crowned by the Danish Royal Academy of Science. Published in *Rendiconti del Circolo Matematico di Palermo*, XXXIV (1912), pp. 41–88. See also the references in this paper.

§ *Mathematische Annalen*, LXXI (1912), pp. 510–532.

‖ *Rendiconti del Circolo Matematico di Palermo*, XXXV (1913), pp. 177–216.

¶ Landau, *loc. cit.*, pp. 192–197. See also the literature cited by Landau here and on p. 154 of the same memoir.

Now, power series (both ascending and descending), factorial series and binomial coefficient series are all included as special cases in the following two types of series of iterated linear fractional functions:

$$\alpha_0+\frac{\alpha_1}{x}+\sum_{n=1}^{\infty}\frac{\alpha_{n+1}}{xS_1(x)S_2(x)\ldots\ldots S_n(x)}, \tag{1}$$

$$\alpha_0+\sum_{n=1}^{\infty}\alpha_n S_1(x)S_2(x)\ldots\ldots S_n(x), \tag{2}$$

where

$$S_1(x)=\frac{ax+b}{cx+d},\quad ad-bc\neq 0;\qquad S_k(x)=\frac{aS_{k-1}(x)+b}{cS_{k-1}(x)+d},\quad k>1,$$

a, b, c, d being constants. We shall refer to these two series as of type I and type II respectively.

If $S_1(x)=ax$, it is clear that (1) is a descending power series and that (2) is an ascending power series. If $S_1(x)=x+1$, then (1) is a factorial series of the form above; while, if $S_1(x)=x-1$, (2) is a binomial coefficient series of the form above.

The object of the present paper is to develop the fundamental elements of a general theory of both the above types of series of iterated linear fractional functions.

In § 1 I introduce some preliminary notations and definitions and state some lemmas which are of frequent use in the convergence proofs.

In § 2 I determine the character of the regions of convergence, of absolute convergence and of conditional convergence of series of both the types I and II. An upper bound to the magnitude of the region of conditional convergence is obtained. Thus we have generalizations of the corresponding theories for the case of power series, factorial series and binomial coefficient series. The methods of Landau for series of the two latter kinds are in the main employed. The general plan of treatment is improved in one respect by the use of the criterion of Gauss for the convergence of series; thus it is no longer necessary to employ properties of the gamma function for factorial series and binomial coefficient series, or of corresponding functions for the other cases (treated for the first time in the present paper). It should be noted here that the general results are in some respects in marked contrast to the simpler ones for the special cases which have been investigated heretofore.

In § 3 the regions of uniform convergence of the series are determined and some immediate consequences are stated.

In § 4 I determine the boundaries of the regions of convergence and absolute convergence of the series in terms of their coefficients.

§ 1. *Preliminary Definitions, Notations and Lemmas.*

In relation to the matter of convergence of series (1) or series (2), it is obvious that an exceptional rôle is played by a point x for which $S_k(x)$ is either zero or infinity for some value of k. In the case of series (1), the point $x=0$ is also exceptional. Later, it will be seen that an exceptional rôle is played by a point x_0 such that $S_1(x_0)=x_0$, whence $S_k(x_0)=x_0$ for every k. Hence we shall employ the following definition:

For series (2) a point x_0 such that $S_k(x_0)=0$, x_0 or ∞, for some k, is called an *exceptional* point. For series (1) these points and the point $x=0$ are said to be *exceptional.* In either case, the remaining points are called *non-exceptional* points.

By the *region of convergence* of series (1) [series (2)] we shall mean that portion of the plane which is made up of those non-exceptional points x at which the series converges and those exceptional points which have the property that there is some neighborhood of each of them such that all points in these neighborhoods (except possibly the exceptional points themselves) are points of convergence of the series. We also define similarly the *region of absolute convergence* of each series.

We shall say that the substitution $x'=S_1(x)$ is the substitution *corresponding* to series (1) or to series (2); also, that these series correspond to the given substitution.

Throughout the paper we shall require to have at hand explicit formulæ for $S_n(x)$ in terms of x and n. In the statement of these formulæ it is convenient to distinguish four cases as follows:*

CASE A. The substitution $x'=S_1(x)$ has two double points in the finite plane. Denote these by α and β. Then for the substitution itself and for the value of $S_n(x)$ we have respectively:

$$\frac{x'-\alpha}{x'-\beta}=k\frac{x-\alpha}{x-\beta}, \quad k\neq 0,\ 1; \qquad S_n(x)\equiv\frac{k^n\beta(x-\alpha)-\alpha(x-\beta)}{k^n(x-\alpha)-(x-\beta)}.$$

CASE B. The substitution $x'=S_1(x)$ has two double points, one of them being at infinity. If the other is at α, then for the substitution and for $S_n(x)$ we have respectively:

$$x'-\alpha=k(x-\alpha), \quad k\neq 0,\ 1; \qquad S_n(x)\equiv k^n(x-\alpha)+\alpha.$$

CASE C. The substitution $x'=S_1(x)$ has only one double point, and this point lies in the finite plane. If it is at α, then for the substitution and for $S_n(x)$ we have respectively:

* As a matter of convenience, the identical substitution is excluded throughout. This involves no loss of generality, since the series corresponding to the substitution $x'=x$, in each type, is of the same form as that corresponding to the more general substitution $x'=ax$, where a is any non-zero constant whatever.

$$\frac{1}{x'-\alpha}=\frac{1}{x-\alpha}+t, \quad t\neq 0; \qquad S_n(x)\equiv\frac{\alpha nt(x-\alpha)+x}{nt(x-\alpha)+1}.$$

CASE D. The substitution $x'=S_1(x)$ has a single double point, and this point is at infinity. Then for the substitution and for $S_n(x)$ we have respectively:

$$x'=x+t, \quad t\neq 0; \qquad S_n(x)\equiv x+nt.$$

It is convenient to note here the behavior of $S_n(x)$ for n approaching infinity. This is simplest in the cases C and D. Let $\bar{x}$ be any finite point which is not a double point of the substitution $x'=S_1(x)$. Then, in cases C and D, it is clear that $\lim_{n=\infty} S_n(\bar{x})$ exists and is the double point of the substitution.

In case A [B], if $\bar{x}$ is any finite point which is not a double point of $x'=S_1(x)$, then $\lim_{n=\infty} S_n(\bar{x})$ exists or not, according as $|k|\neq 1$ or $|k|=1$. In case $|k|>1$, the limit is the double point $\beta[\infty]$. In case $|k|<1$, the limit is the double point $\alpha[\alpha]$. In case $|k|=1$, in which case the limit does not exist, it is clear that $|S_n(\bar{x})|$ is less than some fixed constant for all n greater than some N.

It is convenient to state here the following lemmas which will be useful in several convergence proofs:

LEMMA I. *Let $b_0, b_1, b_2, \ldots.$ and $c_0, c_1, c_2, \ldots.$ be two sequences of complex numbers such that both of the series*

$$\sum_{n=0}^{\infty} b_n, \qquad \sum_{n=0}^{\infty} |c_n-c_{n+1}|$$

are convergent. Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

also converges.

LEMMA II. *Let $b_0, b_1, b_2, \ldots.$ and $c_0, c_1, c_2, \ldots.$ be two sequences of complex numbers such that*

1) *there exists a number B such that $\left|\sum_{n=0}^{t} b_n\right|<B$ for every t;*

2) $\lim_{n=\infty} c_n$ *exists and is zero;*

3) *the series $\sum_{n=0}^{\infty} |c_n-c_{n+1}|$ converges.*

Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

also converges.

LEMMA III. *Let $b_0, b_1, b_2, \ldots.$ be a sequence of complex constants and $c_0, c_1, c_2, \ldots.$ be a sequence of functions of the complex variable x which are*

regular throughout a given domain D (including the boundary). Suppose that

1) *there exists a number B such that* $\left|\sum_{n=0}^{t} b_n\right| < B$ *for every* t;

2) *as n approaches infinity* c_n *converges to* 0 *uniformly in D;*

3) *the series* $\sum_{n=0}^{\infty} |c_n - c_{n+1}|$ *converges uniformly in D.*

Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

converges uniformly in D.

LEMMA IV. *Let* $b_0, b_1, b_2, \ldots$ *be a sequence of complex constants such that the series* $b_0 + b_1 + b_2 + \ldots$ *is convergent. Let* $c_0, c_1, c_2, \ldots$ *be a sequence of functions of the complex variable x which are regular throughout a given domain D (including the boundary) and are such that the series*

$$\sum_{n=0}^{\infty} |c_n - c_{n+1}|$$

converges uniformly in D. Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

is uniformly convergent in D.

Elegant elementary proofs of the first three of these lemmas are given by Landau (*loc. cit.*, pp. 155–157, 160–161); he also gives references to their earlier use. The fourth lemma appears to have been first employed by Nielsen;* but Nielsen's statement of it is not entirely accurate, as Landau† has already pointed out.

§ 2. *Character of the Regions of Convergence and Absolute Convergence.*

By means of lemma I we shall now determine the character of the region of convergence of the series

$$\Omega(x) = \alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{x S_1(x) S_2(x) \ldots S_n(x)}.$$

Let x_0 and x_1 be two non-exceptional values of x for the series. We shall assume that the series above converges for $x = x_0$, and shall express this briefly by saying that $\Omega(x_0)$ converges. We shall determine relations between x_0 and x_1 which are sufficient to ensure that $\Omega(x_1)$ also converges.

Let us put

$$b_n = \frac{\alpha_{n+1}}{x_0 S_1(x_0) S_2(x_0) \ldots S_n(x_0)}, \qquad c_n = \frac{x_0 S_1(x_0) S_2(x_0) \ldots S_n(x_0)}{x_1 S_1(x_1) S_2(x_1) \ldots S_n(x_1)}.$$

* *Annali di matematica*, (3) XV (1908), pp. 275–282.

† *Sitzungsber. d. Kgl. Bayer. Akad. d. Wiss., Math.-Phys.*, 1909.

Then the series $b_1+b_2+b_3+\ldots.$ converges, by hypothesis. From lemma I it follows that the series $b_1c_1+b_2c_2+b_3c_3+\ldots.$, and hence the series $\Omega(x_1)$, converges provided that

$$\sum_{n=1}^{\infty}|c_n-c_{n+1}| \tag{3}$$

is convergent. Here we have

$$c_n-c_{n+1}=\frac{x_0S_1(x_0)\ldots.S_n(x_0)}{x_1S_1(x_1)\ldots.S_n(x_1)}\left(1-\frac{S_{n+1}(x_0)}{S_{n+1}(x_1)}\right).$$

For determining the relation between x_0 and x_1 so that (3) shall converge, we shall have use for the following formulæ, which are easily verified:

$$\left.\begin{aligned}
Q_n(x_0, x_1) &\equiv \frac{S_n(x_0)}{S_n(x_1)}\\
&=1+\frac{k^n(\alpha-\beta)^2(x_0-x_1)}{\{k^n(x_0-\alpha)-(x_0-\beta)\}\{k^n\beta(x_1-\alpha)-\alpha(x_1-\beta)\}}, \text{ in case A;*}\\
&=1+\frac{k^n(x_0-x_1)}{k^n(x_1-\alpha)+\alpha}, \text{ in case B;}\\
&=1+\frac{x_0-x_1}{\{nt(x_0-\alpha)+1\}\{\alpha nt(x_1-\alpha)+x_1\}}, \text{ in case C;}\\
&=1+\frac{x_0-x_1}{x_1+nt}, \text{ in case D.}
\end{aligned}\right\}\tag{4}$$

The ratio $r_n(x_1)$ of the $(n+1)$-th term to the n-th term of series (3) is

$$r_n(x_1)=|Q_{n+1}(x_0, x_1)|\cdot\frac{|1-Q_{n+2}(x_0, x_1)|}{|1-Q_{n+1}(x_0, x_1)|}.$$

Making use of the above values of the function $Q_n(x_0, x_1)$, we have by obvious reductions the following formulæ:

$$\begin{aligned}
r_n(x_1)&=|Q_{n+1}(x_0, x_1)|\cdot\left|k\frac{\{k^{n+1}(x_0-\alpha)-(x_0-\beta)\}\{k^{n+1}\beta(x_1-\alpha)-\alpha(x_1-\beta)\}}{\{k^{n+2}(x_0-\alpha)-(x_0-\beta)\}\{k^{n+2}\beta(x_1-\alpha)-\alpha(x_1-\beta)\}}\right|, \\
&\qquad\text{in case A;}\\
&=|Q_{n+1}(x_0, x_1)|\cdot\left|k\frac{k^{n+1}(x_1-\alpha)+\alpha}{k^{n+2}(x_1-\alpha)+\alpha}\right|, \text{ in case B;}\\
&=|Q_{n+1}(x_0, x_1)|\cdot\left|\frac{\{(n+1)t(x_0-\alpha)+1\}\{(n+1)\alpha t(x_1-\alpha)+x_1\}}{\{(n+2)t(x_0-\alpha)+1\}\{(n+2)\alpha t(x_1-\alpha)+x_1\}}\right|,\\
&\qquad\text{in case C;}\\
&=|Q_{n+1}(x_0, x_1)|\cdot\left|\frac{x_1+(n+1)t}{x_1+(n+2)t}\right|, \text{ in case D.}
\end{aligned}$$

In the discussion immediately following we treat separately the cases A, B, C, D.

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*The definition of the several cases is given in § 1.

In case A, several possibilities arise:

1) If $|k|<1$ and $\alpha\neq 0$, then $\lim\limits_{n=\infty}|Q_n(x_0, x_1)|=1$, and we have
$$\lim_{n=\infty} r_n=|k|.$$

2) If $|k|>1$ and $\beta\neq 0$, then $\lim\limits_{n=\infty}|Q_n(x_0, x_1)|=1$, and we have
$$\lim_{n=\infty} r_n=\left|\frac{1}{k}\right|.$$

3) If $|k|<1$ and $\alpha=0$, we have by easy reductions
$$\lim_{n=\infty} r_n=\lim_{n=\infty}\left|\frac{x_0}{x_1}\right|\cdot\left|\frac{k^n x_1-(x_1-\beta)}{k^n x_0-(x_0-\beta)}\right|\cdot\left|\frac{k^{n+1}x_0-(x_0-\beta)}{k^{n+2}x_0-(x_0-\beta)}\right|=\left|\frac{x_0}{x_1}\right|\cdot\left|\frac{x_1-\beta}{x_0-\beta}\right|.$$

4) If $|k|>1$ and $\beta=0$, we have similarly
$$\lim_{n=\infty} r_n=\left|\frac{x_0}{x_1}\right|\cdot\left|\frac{x_1-\alpha}{x_0-\alpha}\right|.$$

5) If $|k|=1$, we have
$$r_n=\left|\frac{k^{n+2}\beta(x_0-\alpha)-\alpha k(x_0-\beta)}{k^{n+2}(x_0-\alpha)-(x_0-\beta)}\right|\cdot\left|\frac{k^{n+2}(x_1-\alpha)-k(x_1-\beta)}{k^{n+2}\beta(x_1-\alpha)-\alpha(x_1-\beta)}\right|.$$

From these results we conclude that, in cases 1) and 2), $\Omega(x_1)$ converges for every x_1 (subject to the initial conditions specified above). In case 3), $\Omega(x_1)$ converges if
$$\left|\frac{x_1-\beta}{x_1}\right|<\left|\frac{x_0-\beta}{x_0}\right|.$$
In case 4), $\Omega(x_1)$ converges if
$$\left|\frac{x_1-\alpha}{x_1}\right|<\left|\frac{x_0-\alpha}{x_0}\right|.$$
In case 5), $\Omega(x_1)$ converges if $\limsup\limits_{n=\infty} r_n<1$. In several respects this case is exceptional. A corresponding exception arises under case B below; in connection with the latter, an example is given to indicate the nature of the irregularity.

In case B, we have readily
$$r_n=\left|\frac{k^{n+2}(x_0-\alpha)+k\alpha}{k^{n+2}(x_1-\alpha)+\alpha}\right|.$$
Hence, if $\alpha=0$ or if $|k|>1$, we have
$$\lim_{n=\infty} r_n=\left|\frac{x_0-\alpha}{x_1-\alpha}\right|.$$
In this case, series (3), and hence $\Omega(x_1)$, converges if $|x_1-\alpha|>|x_0-\alpha|$. If $\alpha\neq 0$ and $|k|<1$, we have
$$\lim_{n=\infty} r_n=|k|;$$

and therefore, in this case, $\Omega(x_1)$ converges for every x_1 (subject to the initial conditions specified above). If $\alpha \neq 0$ and $|k|=1$, $\Omega(x_1)$ will converge if

$$\limsup_{n=\infty} r_n < 1.$$

This last case is quite exceptional in its character. In our discussion below, we shall point out that the region of convergence (when it is not the entire plane) is always bounded by a circle (or a straight line, considered as a limiting case of a circle) except for cases A and B in which $|k|=1$. By means of examples we shall here show that in this exceptional case the region of convergence may be bounded by a curve of the fourth or even higher degree. Suppose that the substitution is $x'=-x+2$. Then $S_{2m}(x)\equiv x$ and $S_{2m+1}(x) \equiv -x+2$. Consider the series

$$\Omega(x)=1+\frac{1}{x}+\frac{1}{x(x-2)}+\frac{1}{x^2(x-2)}+\frac{1}{x^2(x-2)^2}+\cdots\cdot\cdot$$

If $\bar{x}$ is any non-exceptional value of x, it is easy to show that this series converges if $|\bar{x}(\bar{x}-2)|<1$, and diverges if $|\bar{x}(\bar{x}-2)|>1$. Hence the boundary of its region of convergence is the curve $|x(x-2)|=1$. This is obviously a curve of the fourth degree. By taking a periodic substitution of period greater than 2, we should similarly obtain series whose regions of convergence are bounded by curves of higher degree. Similar examples are readily constructed for the case A, when $|k|=1$. On account of the exceptional character of the cases when $|k|=1$, they will be excluded from further consideration.

We turn now to a further consideration of case C. We have readily

$$r_n=\left|\frac{(\{n+1)\alpha t(x_0-\alpha)+x_0\}\{(n+1)t(x_1-\alpha)+1\}}{(\{n+2)\alpha t(x_1-\alpha)+x_1\}\{(n+2)t(x_0-\alpha)+1\}}\right|.$$

If $\alpha\neq 0$, we have

$$r_n=\left|1-\frac{2}{n}+\cdots\right|,$$

where the terms omitted involve higher powers of $1/n$. Hence

$$r_n=1-\frac{2}{n}+\cdots\cdot\cdot$$

Applying the criterion of Gauss,* we see that series (3), and hence $\Omega(x_1)$, is convergent. If $\alpha=0$, we have

$$r_n=\left|\frac{x_0}{x_1}\cdot\frac{(n+1)tx_1+1}{(n+2)tx_0+1}\right|=\left|1-\frac{1+\frac{1}{t}\left(\frac{1}{x_0}-\frac{1}{x_1}\right)}{n}+\cdots\right|;$$

hence, if we use the notation $R(z)$ for the real part of z, we have

$$r_n=1-\frac{1+R\left(\frac{1}{tx_0}-\frac{1}{tx_1}\right)}{n}+\cdots\cdot\cdot$$

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* See "Encyclopédie des sciences mathématiques," I_1, p. 216.

From this, by aid of the criterion of Gauss, we conclude that series (3), and hence $\Omega(x_1)$, converges if

$$R\left(\frac{1}{tx_1}\right) < R\left(\frac{1}{tx_0}\right).$$

Finally, let us consider case D. Here we have

$$r_n = \left|\frac{x_0+(n+1)t}{x_1+(n+2)t}\right| = 1 - \frac{1+R\left(\frac{x_1-x_0}{t}\right)}{n} + \ldots.$$

Using again the criterion of Gauss, we conclude that series (3), and hence $\Omega(x_1)$, converges if

$$R\left(\frac{x_1}{t}\right) > R\left(\frac{x_0}{t}\right).$$

In the discussion above we have noted that cases A and B are exceptional if $|k|=1$. The results for the other cases may be stated in the following theorem:

THEOREM I_1.* *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$\Omega(x) = \alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x)\ldots.S_n(x)},$$

and suppose that the series converges for $x=x_0$. Then the series converges in the following cases:

CASE A.† *If $|k|<1$ and $\alpha \neq 0$, or if $|k|>1$ and $\beta \neq 0$, then $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|<1$ and $\alpha=0$, $\Omega(x_1)$ converges if*

$$\left|\frac{x_1-\beta}{x_1}\right| < \left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $\Omega(x_1)$ converges if

$$\left|\frac{x_1-\alpha}{x_1}\right| < \left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|<1$ and $\alpha \neq 0$, $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|>1$, or if $\alpha=0$, then $\Omega(x_1)$ converges if*

$$|x_1-\alpha| > |x_0-\alpha|.$$

CASE C. *If $\alpha \neq 0$, $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $\alpha=0$, $\Omega(x_1)$ converges if*

$$R\left(\frac{1}{tx_1}\right) < R\left(\frac{1}{tx_0}\right).$$

CASE D. *In this case, $\Omega(x_1)$ converges if*

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*The subscript 1 [2] attached to the number of a theorem indicates that the theorem refers to a series of type I [II]. See the Introduction for definition of types.

† See the definition of cases A, B, C, D in § 1.

$$R\left(\frac{x_1}{t}\right) > R\left(\frac{x_0}{t}\right).$$

If one employs lemma II instead of lemma I, the above theorem can in some cases be slightly strengthened by requiring, in the hypothesis, that the sum of t terms of $\Omega(x_0)$ shall be bounded in absolute value, instead of making the stronger assumption that this series converges. To prove this, it is sufficient to show further that c_n approaches the limit zero as n increases indefinitely. This is equivalent to showing that the infinite product

$$\prod_{n=1}^{\infty} \frac{S_n(x_0)}{S_n(x_1)}$$

diverges to zero when x_0 and x_1 are connected by the relations given in the theorem. If we make use of equations (4) we see readily that this product is zero in case A if $|k|<1$ and $\alpha=0$ or if $|k|>1$ and $\beta=0$, in case B if $|k|>1$ or if $\alpha=0$, in case C if $\alpha=0$, in case D for all t. Hence, in these cases the theorem may be strengthened as indicated.

It is not difficult to construct examples in which $\Omega(x)$ converges for no non-exceptional value of x whatever. For the moment, we exclude this case from consideration. Then the region of convergence* of $\Omega(x)$ is the entire plane in case A if $|k|>1$ and $\beta \neq 0$, in cases A and B if $|k|<1$ and $\alpha \neq 0$, in case C if $\alpha \neq 0$. In all other cases the region of convergence may or may not be the entire plane, the fact in a particular case depending on the coefficients α of the series. We shall now take up these remaining cases separately and determine the exact nature of the region of convergence when it is not the entire plane.

The results are simple in case D, and hence we shall treat this first. Consider the straight line $(0t)$ through the points 0 and t. If $\Omega(x)$ converges at any non-exceptional point $\bar{x}$ on this line, it converges at every non-exceptional point to the right of a line through $\bar{x}$ and perpendicular to $(0t)$, the directions right and left being determined by saying that t is to the right of 0. From this it follows readily that there exists a straight line l perpendicular to $(0t)$ such that $\Omega(x)$ converges for every non-exceptional x to the right of l and diverges for every non-exceptional x to the left of l. On l, its character as to convergence or divergence varies as in the case of a power series in relation to its circle of convergence, as one might show by examples. The line l is called the *line of convergence* of the series. We may look upon l as a circle through the double point ∞ of the substitution $x'=x+t$ corresponding to the series $\Omega(x)$.

*See the definition of region of convergence in § 1.

In case C, when $\alpha=0$, the results are analogous to those in case D, as we shall now show. Here the double point of the substitution corresponding to $\Omega(x)$ is 0. Consider the system S of circles C which pass through the point 0 and have their centers on the straight line $(0t)$ through the points 0 and t. We shall say that t is to the right of 0. We make the following conventions concerning the *interior* and *exterior* of these circles C: If C lies to the right of the straight line l through 0 and perpendicular to $(0t)$, we shall say that the *interior* of the circle is that part of the plane bounded by C and containing its center; if C coincides with l, its interior is to the right of l; if C is to the left of l, then its interior is that part of the plane bounded by C and not containing its center. Now if $\Omega(x)$ converges for any non-exceptional point on a given circle C of the system S, it follows from the above theorem that it converges at every non-exceptional point *exterior* to C. From this it is readily seen that there exists a circle $\bar{C}$ of this system S such that $\Omega(x)$ converges for every non-exceptional point *exterior* to $\bar{C}$ and diverges for every non-exceptional point *interior* to $\bar{C}$. At a point on $\bar{C}$ it may either converge or diverge, as one might show by examples. The circle $\bar{C}$ will be referred to as the *circle of convergence* of $\Omega(x)$.

Under B we have two cases to examine; namely, that when $\alpha=0$ and that when $|k|>1$. In either case the above theorem leads readily to the conclusion that there exists a circle $\bar{C}$ about α as a center such that $\Omega(x)$ converges for every non-exceptional point x *exterior* to $\bar{C}$ and diverges for every non-exceptional point *interior* to $\bar{C}$, the words exterior and interior being now employed in their usual (elementary) sense. The circle $\bar{C}$ is called the *circle of convergence* of $\Omega(x)$.

Finally, under A we have also two cases to consider; namely, that when $|k|<1$ and $\alpha=0$ and that when $|k|>1$ and $\beta=0$. On account of the similarity of these two cases, it is sufficient to treat in detail one of them alone; we take that when $|k|<1$ and $\alpha=0$. Consider the systems S of circles C such that a circle C of the system is defined by the property that the distances of a point P on C from the points 0 and β is a constant for all positions of P. This system contains as a particular case the straight line which bisects perpendicularly the straight line segment joining 0 and β. For every circle C the points 0 and β are on opposite sides of C. The part of the plane bounded by C and containing $0\,[\beta]$ will be called the *interior* [*exterior*] of the circle. It is now easy to show, by aid of the above theorem, that there exists a circle $\bar{C}$ of the system S such that $\Omega(x)$ converges at every non-exceptional point *exterior* to $\bar{C}$ and diverges at every non-exceptional point *interior* to $\bar{C}$. The circle $\bar{C}$ is called the *circle of convergence* of $\Omega(x)$.

Corresponding to theorem I_1 for series of type I, we have the following theorem I_2 for series of type II:

THEOREM I_2. *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$W(x)=\alpha_0+\sum_{n=1}^{\infty}\alpha_n S_1(x)S_2(x)\ldots.S_n(x),$$

and suppose that the series converges for $x=x_0$. Then the series $W(x_1)$ converges in the following cases:

CASE A. *If $|k|<1$ and $\alpha\neq 0$, or if $|k|>1$ and $\beta\neq 0$, then $W(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|<1$ and $\alpha=0$, $W(x_1)$ converges if*

$$\left|\frac{x_1-\beta}{x_1}\right|>\left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $W(x_1)$ converges if

$$\left|\frac{x_1-\alpha}{x_1}\right|>\left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|<1$ and $\alpha\neq 0$, $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|>1$, or if $\alpha=0$, then $W(x_1)$ converges if*

$$|x_1-\alpha|<|x_0-\alpha|.$$

CASE C. *If $\alpha\neq 0$, $W(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $\alpha=0$, $W(x_1)$ converges if*

$$R\left(\frac{1}{tx_1}\right)>R\left(\frac{1}{tx_0}\right).$$

CASE D. *In this case, $W(x_1)$ converges if*

$$R\left(\frac{x_1}{t}\right)<R\left(\frac{x_0}{t}\right).$$

For the proof of this theorem also we employ lemma I. We write

$$b_n=\alpha_n S_1(x_0)S_2(x_0)\ldots.S_n(x_0),\qquad c_n=\frac{S_1(x_1)S_2(x_1)\ldots.S_n(x_1)}{S_1(x_0)S_2(x_0)\ldots.S_n(x_0)}.$$

Then the series $b_1+b_2+b_3+\ldots.$ converges, by hypothesis. From lemma I it follows that the series $b_1c_1+b_2c_2+b_3c_3+\ldots.$, and hence $W(x_1)$, converges, provided that

$$\sum_{n=1}^{\infty}|c_n-c_{n+1}|$$

is convergent. Now this series may be obtained from (3) by multiplying each term of series (3) by $|x_1/x_0|$ and in the result exchanging x_0 and x_1. Hence, we have only to repeat the argument for theorem I_1, interchanging the rôles of x_0 and x_1, in order to complete the proof of theorem I_2.

By the use of lemma II instead of lemma I, it is possible to strengthen this theorem by a slight weakening of the hypothesis. The discussion is analogous to that for theorem I_1; it is omitted here.

It is clear that the present theorem enables us to determine completely the character of the region of convergence for $W(x)$. It may consist of the entire plane or it may be non-existent, in special cases. Laying aside these possibilities, it is easy to see that the boundary of the region of convergence, in every case, must be of the same character for $W(x)$ as for $\Omega(x)$. In each case, however, the region of convergence for $W(x)$ is on the opposite side of the boundary from that for $\Omega(x)$. This is analogous to the corresponding facts for ascending and descending power series. This is natural, since $W(x)$ contains ascending power series as a special case, while $\Omega(x)$ contains descending power series as a special case.

We turn now to a consideration of the character of the region of absolute convergence of $\Omega(x)$. Let us suppose that $\Omega(x_0)$ is absolutely convergent, and ascertain conditions which are sufficient to ensure that $\Omega(x_1)$ is absolutely convergent. One such condition is that the ratio

$$c_n = \frac{x_0 S_1(x_0) \dots . S_n(x_0)}{x_1 S_1(x_1) \dots . S_n(x_1)} \tag{5}$$

of corresponding terms of $\Omega(x_1)$ and $\Omega(x_0)$ is bounded in absolute value. Clearly this ratio is bounded if the infinite product

$$\prod_{n=1}^{\infty} \left| \frac{S_n(x_0)}{S_n(x_1)} \right|$$

is convergent or if it diverges to zero. For the study of this matter we may make the same separation into cases as in theorem I_1. By taking up each case separately, making use of equations (4), and applying elementary tests of convergence, or divergence to zero, of an infinite product, one may easily show that $|c_n|$ is bounded, in each case, provided that x_0 and x_1 satisfy the relations specified in theorem I_1 for such case. Hence we have the following result:

THEOREM II_1. *If throughout theorem* I_1 *we replace the word "converges" by the words "converges absolutely," we obtain a new theorem which is valid.*

By a precisely similar argument one may also prove the following theorem:

THEOREM II_2. *If throughout theorem* I_2 *we replace the word "converges" by the words "converges absolutely," we obtain a new theorem which is valid.*

By a discussion in all respects similar to that which follows theorem I_1, it is now possible to determine completely the character of the regions of absolute convergence of both $\Omega(x)$ and $W(x)$. It is clear that in each case this region is of the same character as the region of convergence for the corresponding case. Consequently, it is unnecessary to go into the treatment in detail.

In view of the preceding discussion, it is an easy matter to construct series having the following interesting property: They converge for every non-exceptional value of x; they converge absolutely for no non-exceptional value of x. We shall illustrate this remark by a single example.

In case A, put $\alpha=0$, $\beta=1$, $k=2$. Then we have

$$S_n(x)=\frac{2^n x}{(2^n-1)x+1}.$$

Consider the series

$$\Omega(x)=\sum_{n=1}^{\infty}\frac{\alpha_{n+1}}{xS_1(x)\ldots .S_n(x)},$$

where

$$\alpha_{n+1}=\frac{(-1)^n}{n}\prod_{k=1}^{n}\frac{2^{k+1}}{2^{k+1}-1}.$$

For the non-exceptional value $x=2$, this series converges, but it does not converge absolutely. Hence, from theorems I_1 and II_1 (and an examination of the series for exceptional values of x) it follows that $\Omega(x)$ converges for every x different from zero; it converges absolutely, however, only for the exceptional values

$$x=-\frac{1}{2^n-1},\qquad n=1,\,2,\,3,\,\ldots .;$$

and then trivially, since for each of these values it has only a finite number of terms different from zero.

This example raises the general question as to the character of the region of conditional convergence of the series $\Omega(x)$ and $W(x)$. It is easy to see, in the light of the preceding theorems, and for the case of a series of each type, that the region of conditional convergence may be non-existent (through the series either diverging everywhere or converging absolutely throughout its region of convergence) or may be the entire region of convergence, in the following cases:

Case A: $|k|<1$ and $\alpha\neq 0$; $|k|>1$ and $\beta\neq 0$.

Case B: $|k|<1$ and $\alpha\neq 0$.

Case C: $\alpha\neq 0$.

To investigate the remaining cases we proceed as follows: Let x_0 and x_1 be non-exceptional values for the series $\Omega(x)$. Suppose that there exists a constant A such that the n-th term of $\Omega(x_0)$ is in absolute value less than A for every n. Denote by c_n the ratio (given in (5)) between corresponding terms of $\Omega(x_0)$ and $\Omega(x_1)$. Then $\Omega(x_1)$ converges absolutely if

$$|c_1|+|c_2|+|c_3|+\ldots .$$

converges. Now the ratio $|c_n|/|c_{n-1}|$ of two consecutive terms of this series

is $|Q_n(x_0, x_1)|$. If, now, in investigating the convergence of this series, we employ equations (4) and make use of the simple ratio test and the Gauss criterion for convergence of series, we are led to the following theorem:

THEOREM III$_1$. *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$\Omega(x) = \alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x)\ldots S_n(x)},$$

and suppose that there exists a constant A such that the n-th term of $\Omega(x_0)$ is in absolute value less than A for every n. Then $\Omega(x_1)$ converges absolutely in the following cases:

CASE A. *If $|k|<1$ and $\alpha \doteq 0$, $\Omega(x_1)$ converges absolutely if*

$$\left|\frac{x_1-\beta}{x_1}\right| < \left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $\Omega(x_1)$ converges absolutely if

$$\left|\frac{x_1-\alpha}{x_1}\right| < \left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|>1$, or if $\alpha=0$, then $\Omega(x_1)$ converges absolutely if*

$$|x_1-\alpha| > |x_0-\alpha|.$$

CASE C. *If $\alpha=0$, $\Omega(x_1)$ converges absolutely if*

$$R\left(\frac{1}{tx_0}\right) - R\left(\frac{1}{tx_1}\right) > 1.$$

CASE D. *In this case, $\Omega(x_1)$ converges absolutely if*

$$R\left(\frac{x_1}{t}\right) - R\left(\frac{x_0}{t}\right) > 1.$$

In a similar way one readily proves the following theorem:

THEOREM III$_2$. *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$W(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n S_1(x)S_2(x)\ldots S_n(x),$$

and suppose that there exists a constant A such that the n-th term of $W(x_0)$ is in absolute value less than A for every n. Then $W(x_1)$ converges absolutely in the following cases:

CASE A. *If $|k|<1$ and $\alpha=0$, $W(x_1)$ converges absolutely if*

$$\left|\frac{x_1-\beta}{x_1}\right| > \left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $W(x_1)$ converges absolutely if

$$\left|\frac{x_1-\alpha}{x_1}\right| > \left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|>1$, or if $\alpha=0$, then $W(x_1)$ converges absolutely if*

$$|x_1-\alpha| < |x_0-\alpha|.$$

CASE C. *If $\alpha=0$, $W(x_1)$ converges absolutely if*

$$R\left(\frac{1}{tx_1}\right)-R\left(\frac{1}{tx_0}\right)>1.$$

CASE D. *In this case, $W(x_1)$ converges absolutely if*

$$R\left(\frac{x_0}{t}\right)-R\left(\frac{x_1}{t}\right)>1.$$

From the last two theorems it follows that $\Omega(x)$ and $W(x)$ both converge absolutely at every non-exceptional point in the interior of their regions of convergence, in the following cases: Case A, when $|k|>1$ and $\beta=0$; cases A and B, when $|k|<1$ and $\alpha=0$. Here the general theory is analogous to that of power series, as indeed it should be, since in these cases the series are direct generalizations of power series.

In case D, the region of conditional convergence is a strip bounded by the line of convergence and the line of absolute convergence, the width of this strip being at most $|t|$. It is possible to construct examples (cf. Landau, *loc. cit.*) to show that each of the logical possibilities may arise; namely, that the strip of conditional convergence is non-existent, that it exists and is of width less than $|t|$, that it is of width $|t|$.

In case C, when $\alpha=0$, it is easy to see that there may again exist a region of conditional convergence. It is bounded by two circles, each passing through the point 0 and having its center on the straight line $(0t)$ through the points 0 and t. The maximum distance between the centers of these two circles—the circle of convergence and the circle of absolute convergence—is determined in the case of $\Omega(x)$ $[W(x)]$ by means of the relation

$$R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx_1}\right)\leqq 1 \qquad \left[R\left(\frac{1}{tx_1}\right)-R\left(\frac{1}{tx_0}\right)\leqq 1\right],$$

where x_0 is on the circle of convergence and x_1 is on the circle of absolute convergence. The greatest possible width of the region of conditional convergence is thus a function of the radius of convergence of the series.

Examples might be constructed to show (for this case as well as for the preceding) that each of the three logical possibilities may actually arise. This may also be seen indirectly by observing that $\Omega(x)$ $[W(x)]$, for case C when $\alpha=0$, transforms into $W(x-t)$ $[\Omega(x+t)]$ for case D by replacing x by $1/x$.

§ 3. *Uniform Convergence of the Series. Nature of the Functions Defined by Them.*

We shall now prove the following theorem:

THEOREM IV. *The series* $\Omega(x)$ $[W(x)]$ converges uniformly throughout any closed domain D lying within its region of convergence and containing no point x which is exceptional for $\Omega(x)$ $[W(x)]$.*

*The exceptional subdivisions of cases A and B in which $|k|$ is unity are naturally excluded here.

Let us consider first the case of the series $\Omega(x)$. Suppose that x_0 is a non-exceptional value of x for which $\Omega(x)$ is convergent; later we shall subject x_0 to such further conditions as will serve our convenience. Write

$$b_n = \frac{a_{n+1}}{x_0 S_1(x_0) \dots S_n(x_0)}, \qquad c_n = \frac{x_0 S_1(x_0) \dots S_n(x_0)}{x_1 S_1(x_1) \dots S_n(x_1)}.$$

Then the series $b_1 + b_2 + b_3 + \dots$ converges, by hypothesis. From lemma IV it follows that the series $b_1 c_1 + b_2 c_2 + b_3 c_3 + \dots$, and hence the series $\Omega(x)$, converges uniformly throughout D, provided that

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}| \tag{6}$$

converges uniformly in D.

Now series (6) is what series (3) becomes on replacing x_1 by x; and therefore we may employ the results of the reckoning in connection with (3) in the proof of the uniform convergence of (6). Thus we see that, when n increases indefinitely, the ratio $r_n(x)$ of two consecutive terms of (6) approaches a limit which is less than unity (whatever x_0 may be) in each of the following cases: Case A, when $|k| < 1$ and $\alpha \neq 0$ or when $|k| > 1$ and $\beta \neq 0$; case B, when $|k| < 1$ and $\alpha \neq 0$. Hence it is easy to construct a comparison series of constant terms such that series (6) is term by term less than this comparison series. Hence, in these cases (6) is uniformly convergent. In case C, when $\alpha \neq 0$, it is also unnecessary to place any restriction on x_0, since in this case

$$r_n(x) = 1 - \frac{2}{n} + \dots.$$

For, as a comparison series, one may employ a series of the form

$$\sum_{n=1}^{\infty} \frac{A}{n^{3/2}},$$

where A is a properly chosen constant.

In each of the other cases it is necessary to place further restrictions on x_0.

In case A, when $|k| < 1$ and $\alpha = 0$, we have

$$\lim_{n=\infty} r_n(x) = \left| \frac{x_0}{x} \right| \cdot \left| \frac{x - \beta}{x_0 - \beta} \right|.$$

Since x is in D and D lies within the region of convergence for $\Omega(x)$, it is clear that x_0 can be chosen so that the above limit is less than a properly chosen positive constant ρ $(\rho < 1)$ for every x in D. Hence, by means of a comparison series of constant terms of the form

$$A + A\rho + A\rho^2 + \dots,$$

it may be shown in this case also that (6) is uniformly convergent in D. Sim-

ilarly, one may deal with each of the following: Case A, when $|k|>1$ and $\beta=0$; case B, when $\alpha=0$ or $|k|>1$.

In case C, if $\alpha=0$ we have

$$r_n(x)=1-\frac{1+R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx}\right)}{n}+\ldots.$$

Since x is in D and D lies within the region of convergence of $\Omega(x)$, it is easy to see that there exists a positive constant 2ε such that

$$R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx}\right)\geqq 2\varepsilon$$

for every x in D. Therefore, for the proof in this case that (6) converges uniformly in D, it is sufficient to construct a comparison series of the form

$$\sum_{n=1}^{\infty}\frac{A}{n^{1+\epsilon}},$$

where A is a properly chosen positive constant. Similarly, one may deal with case D.

This completes the examination of all the cases for the series $\Omega(x)$; and consequently the theorem is established for this series.

It is unnecessary to give in detail the argument for $W(x)$, since it is so far similar to that for $\Omega(x)$. It is sufficient to point out that one employs lemma IV, using for b_n and c_n the quantities

$$b_n=\alpha_n S_1(x_0)S_2(x_0)\ldots.S_n(x_0),\qquad c_n=\frac{S_1(x)S_2(x)\ldots.S_n(x)}{S_1(x_0)S_2(x_0)\ldots.S_n(x_0)}.$$

For our case D (the case of factorial series and binomial coefficient series), Landau (*loc. cit.*) effects the proof of theorem IV by the use of lemma III. This lemma is obviously not sufficient for the more general case treated here, since it is not always true that c_n approaches zero as n increases indefinitely.

Now, if we make use of the well-known theorem of Weierstrass relative to the analytic character of the function represented as a uniformly convergent series of analytic functions, we are led (in view of theorem IV) to the following theorem:

THEOREM V. *The series* $\Omega(x)$ $[W(x)]$ (*at least if* $|k|$ *is different from unity in cases* A *and* B) *represents a function which is analytic at every non-exceptional point* x *lying within its region of convergence.*

Certain exceptional points for the series are always regular points of the functions represented by them. Thus, a point which is exceptional for one of the series only through causing every term past a certain one in that series to vanish, is clearly a regular point for the function represented by the series, provided that it lies within the region of convergence of the series.

Furthermore, it is easy to show that a point (within the region of convergence of the series) which is exceptional only through causing every term of the series past a certain one to have a pole of the first order at the exceptional point, is either a regular point or a pole of the first order for the function represented by the series (compare Landau, *loc. cit.*, p. 164, where a discussion of this matter for factorial series is given).

§ 4. *Dependence of the Region of Convergence on the Coefficients of the Series.*

In the preceding discussion of the nature of the regions of convergence of the series $\Omega(x)$ and $W(x)$ we have employed no properties of the (constant) coefficients except what is involved in the assumption of convergence or of absolute convergence for some value x_0 of x. We turn now to the question as to how the *magnitude* of the region of convergence depends on the actual coefficients of a given series.

In certain cases this question is trivial; namely, in those cases in which it is true that the series always converges [converges absolutely] for every non-exceptional x as soon as it converges [converges absolutely] for a single non-exceptional x. All that is necessary, in such a case, for a complete answer to the question is to determine the divergence or the nature of the convergence of the series for a single value of x, and then apply the general theorems of § 2.

Our general question here has already been answered by Landau (*loc. cit.*) for the case of factorial series and binomial coefficient series. This obviously affords also the answer to the question for our cases D, since these obviously go over into the simpler cases (treated by Landau) by multiplicative transformations on x.

We have observed above that the simple transformation of replacing x by $1/x$ carries our $\Omega(x)$ [$W(x)$] for case C when $\alpha=0$ over into our $W(x-t)$ [$\Omega(x+t)$] for case D. Hence Landau's theory affords an immediate means for the resolution of the present problem for case C when $\alpha=0$.

For case B when $\alpha=0$ the series $\Omega(x)$ and $W(x)$ are both power series; and hence the problem has been solved for this case.

There remains for further consideration essentially two cases: Namely, case A when $|k|<1$ and $\alpha=0$ (this being equivalent to case A when $|k|>1$ and $\beta=0$) and case B when $|k|>1$. These we shall now take up in turn. Since in each of them the boundary of the region of convergence coincides with that of the region of absolute convergence it is sufficient to treat only the former.

Let us first consider case A when $|k|<1$ and $\alpha=0$. We have

$$S_n(x)\equiv\frac{k^n\beta x}{(k^n-1)x+\beta}, \qquad \beta\neq 0, \quad k\neq 0, \quad |k|<1. \tag{7}$$

The boundary of the region of convergence is a circle whose equation is of the form

$$\left|\frac{x-\beta}{x}\right|=\rho, \tag{8}$$

where ρ is a constant depending on the coefficients $a_1, a_2, \ldots$ of the series. We have to determine ρ. We first prove the following theorem:

THEOREM VI$'_1$. *The two series**

$$\Omega(x)=a+\frac{a_0}{x}+\sum_{n=1}^{\infty}\frac{a_n k^{1+2+\cdots+n}(-\beta)^n}{xS_1(x)S_2(x)\ldots S_n(x)}, \qquad \psi(x)=\sum_{n=1}^{\infty}a_n\left(\frac{x-\beta}{x}\right)^n,$$

in which $S_n(x)$ has the value given in (7), *both converge or both diverge for any given value of x which is non-exceptional for $\Omega(x)$.*

The proof falls into two parts.

1. Suppose that $\psi(x)$ converges for a value x_0 of x which is non-exceptional for $\Omega(x)$. We shall prove that $\Omega(x_0)$ is convergent. Write

$$b_n=a_n\left(\frac{x_0-\beta}{x_0}\right)^n, \qquad c_n=\frac{k^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}\left(\frac{x_0}{x_0-\beta}\right)^n.$$

The series $b_1+b_2+\ldots$ converges, by hypothesis. Hence, from lemma I, it follows that $\Omega(x_0)$ converges, provided that

$$\sum_{n=1}^{\infty}|c_n-c_{n+1}| \tag{9}$$

is convergent.

We have

$$\begin{aligned} c_n-c_{n+1}&=\frac{k^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}\left(\frac{x_0}{x_0-\beta}\right)^n\left(1-\frac{k^{n+1}(-\beta)}{S_{n+1}(x_0)}\cdot\frac{x_0}{x_0-\beta}\right)\\ &=\frac{k^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}\left(\frac{x_0}{x_0-\beta}\right)^n\left(\frac{k^{n+1}x_0}{x_0-\beta}\right).\end{aligned}$$

Thence the ratio r_n of the $(n+1)$-th term to the n-th term of (9) is easily reduced to the form

$$r_n=\left|\frac{k\{(1-k^{n+1})x_0-\beta\}}{x_0-\beta}\right|.$$

Therefore, $\lim_{n=\infty} r_n=|k|$; this being less than unity, it follows that (9), and hence $\Omega(x_0)$, converges.

2. Suppose that $\Omega(x)$ converges for a non-exceptional value x_0 of x. In order to prove that $\psi(x_0)$ converges, it is sufficient to write

$$b_n=\frac{a_nk^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}, \qquad c_n=\frac{x_0S_1(x_0)\ldots S_n(x_0)}{k^{1+2+\cdots+n}(-\beta)^n}\left(\frac{x_0-\beta}{x_0}\right)^n,$$

and apply lemma I. It is not necessary to give the argument in detail.

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*It is convenient here to write the (constant) coefficients in $\Omega(x)$ in a new form. Obviously, there is no loss of generality in this.

From the theorem just demonstrated it follows that the regions of convergence of $\Omega(x)$ and $\psi(x)$ coincide. But $\psi(x)$ is a power series in z, $z=(x-\beta)/x$, and therefore its region of convergence in terms of the coefficients a_n is known. It is the circle (8), where

$$\frac{1}{\rho}=\limsup_{n=\infty}\sqrt[n]{|a_n|}.$$

Hence:

THEOREM VII$'_1$. *The series* $\Omega(x)$ *of theorem* VI$'_1$ *has for the boundary of its region of convergence the circle*

$$\left|\frac{x-\beta}{x}\right|=\rho,$$

where

$$\frac{1}{\rho}=\limsup_{n=\infty}\sqrt[n]{|a_n|}.$$

In a similar manner one may readily prove the following two theorems:

THEOREM VI$'_2$ *The two series*

$$W(x)=a_0+\sum_{n=1}^{\infty}a_n\frac{S_1(x)S_2(x)\ldots.S_n(x)}{k^{1+2+\ldots.+n}(-\beta)^n},\qquad \phi(x)=\sum_{n=1}^{\infty}a_n\left(\frac{x}{x-\beta}\right)^n,$$

in which $S_n(x)$ *has the value given in* (7), *both converge or both diverge for any given value of* x *which is non-exceptional for* $W(x)$.

THEOREM VII$'_2$. *The series* $W(x)$ *of theorem* VI$'_2$ *has for the boundary of its region of convergence the circle*

$$\left|\frac{x}{x-\beta}\right|=\rho,$$

where

$$\frac{1}{\rho}=\limsup_{n=\infty}\sqrt[n]{|a_n|}.$$

We shall next treat the case B when $|k|>1$. We have

$$S_n(x)=k^n(x-\alpha)+\alpha,\qquad |k|>1.$$

We shall first prove the following theorem:

THEOREM VI$''_1$. *The two series*

$$\Omega(x)=a+\frac{a_0}{x}+\sum_{n=1}^{\infty}\frac{a_n}{x}\prod_{t=1}^{n}\frac{k^t}{k^t(x-\alpha)+\alpha},\quad |k|>1;\qquad \psi(x)=\sum_{n=1}^{\infty}\frac{a_n}{(x-\alpha)^n}$$

both converge or both diverge for any given value of x *which is non-exceptional for* $\Omega(x)$.

The proof falls into two parts.

1. Suppose that $\psi(x)$ converges for a value x_0 of x which is non-exceptional for $\Omega(x)$. We shall prove that $\Omega(x_0)$ is also convergent. We put

$$b_n=\frac{a_n}{(x_0-\alpha)^n},\qquad c_n=\frac{1}{x_0}\prod_{t=1}^{n}\frac{k^t}{k^t(x_0-\alpha)+\alpha}\cdot(x_0-\alpha)^n$$

and apply lemma I. In order to show that $\Omega(x_0)$ converges, it is sufficient to prove the convergence of

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}|. \tag{10}$$

We have

$$c_n - c_{n+1} = \frac{1}{x_0} \prod_{t=1}^{n} \frac{k^t}{k^t(x_0-\alpha)+\alpha} \cdot (x_0-\alpha)^n \left(1 - \frac{k^{n+1}(x_0-\alpha)}{k^{n+1}(x_0-\alpha)+\alpha}\right)$$
$$= \frac{1}{x_0} \prod_{t=1}^{n} \frac{k^t}{k^t(x_0-\alpha)+\alpha} \cdot (x_0-\alpha)^n \left(\frac{\alpha}{k^{n+1}(x_0-\alpha)+\alpha}\right).$$

Hence, for the ratio r_n of the $(n+1)$-th term to the n-th term of (10), we have readily

$$r_n = \left|\frac{k^{n+1}(x_0-\alpha)}{k^{n+2}(x_0-\alpha)+\alpha}\right|.$$

Therefore, $\lim\limits_{n=\infty} r_n = |1/k|$; this being less than unity, it follows that (10), and hence $\Omega(x_0)$, converges.

2. Suppose that $\Omega(x_0)$ converges, x_0 being a non-exceptional value of x for $\Omega(x)$. In order to show that $\psi(x_0)$ also converges, it is sufficient to put

$$b_n = \frac{a_n}{x_0} \prod_{t=1}^{n} \frac{k^t}{k^t(x_0-\alpha)+\alpha}, \qquad c_n = x_0 \prod_{t=1}^{n} \frac{k^t(x_0-\alpha)+\alpha}{k^t} \cdot \frac{1}{(x_0-\alpha)^n}$$

and apply lemma I. It is unnecessary to give the argument in detail.

From the preceding theorem we have at once the following:

THEOREM VII$_1''$. *The series $\Omega(x)$ of theorem VI$_1''$ has for the boundary of its region of convergence a circle about α as center with radius ρ, where*

$$\rho = \limsup_{n=\infty} \sqrt[n]{|a_n|}.$$

In a similar manner we may readily prove the following two theorems:

THEOREM VI$_2''$. *The two series*

$$W(x) = a_0 + \sum_{n=1}^{\infty} a_n \prod_{t=1}^{n} \frac{k^t(x-\alpha)+\alpha}{k^t}, \quad |k|>1; \qquad \phi(x) = \sum_{n=1}^{\infty} a_n (x-\alpha)^n$$

both converge or both diverge for any given value of x which is non-exceptional for $W(x)$.

THEOREM VII$_2''$. *The series $W(x)$ of theorem VI$_2''$ has for the boundary of its region of convergence a circle about α as center with radius ρ, where*

$$\frac{1}{\rho} = \limsup_{n=\infty} \sqrt[n]{|a_n|}.$$

INDIANA UNIVERSITY, *July*, 1913.